Einstein - Weyl structures and Bianchi metrics

Guy Bonneau*

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Abstract

We analyse in a systematic way the (non-)compact four dimensional Einstein-Weyl spaces equipped with a Bianchi metric. We show that Einstein-Weyl structures with a Class A Bianchi metric have a conformal scalar curvature of constant sign on the manifold. Moreover, we prove that most of them are conformally Einstein or conformally Kähler ; in the non-exact Einstein-Weyl case with a Bianchi metric of the type VII_0, $VIII$ or IX, we show that the distance may be taken in a diagonal form and we obtain its explicit 4-parameters expression. This extends our previous analysis, limited to the diagonal, Kähler Bianchi IX case.

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*Laboratoire de Physique Théorique et des Hautes Energies, Unité associée au CNRS URA 280, Université Paris 7, 2 Place Jussieu, 75251 Paris Cedex 05. bonneau@lpthe.jussieu.fr

1 Introduction

In the last years, Einstein-Weyl geometry has raised some interest, in particular when in a recent paper, Tod [1] exhibits the relationship between a particular Einstein-Weyl geometry without torsion (the four-dimensional self-dual Einstein-Weyl geometry studied by Pedersen and Swann [2]) and local heterotic geometry (i.e. the Riemannian geometry with torsion and three complex structures, associated with (4,0) supersymmetric non-linear σ models [3, 4, 5]).

To extend these ideas to other situations, we analysed in a first step [6](hereafter referred to as [GB]) Einstein-Weyl equations in the subclass of diagonal Kähler Bianchi IX metrics (in the standard classification [7, 8]). In the present work, we study (non-)compact 4-dimensional Einstein-Weyl structures (for recent reviews see refs. [2, 9]) on cohomogenity-one manifolds with a 3 dimensional group of isometries transitive on codimension-one surfaces, *i.e.*, in the general relativity terminology, Bianchi metrics, and neither require a diagonal metric nor the Kähler property ; we however obtain interesting results for any (class A) Bianchi metrics.

Let us recall that, in the compact case, on general grounds, strong results on Einstein-Weyl structures have been known for some time :

- There exits a unique metric in a given conformal class [g] such that the Weyl form is co-closed [10] ,
$$\nabla_\mu \gamma^\mu = 0 \ .$$
One then speaks of the "Gauduchon's gauge" and of a "Gauduchon's metric".

- The analysis of Einstein-Weyl equations in this gauge gives two essential results :
  - The dual of the Weyl form γ is a Killing vector [11]:
  $$\nabla_{(\mu}\gamma_{\nu)} = 0 \ ,$$
  - Four dimensional Einstein-Weyl spaces have a constant conformal scalar curvature [12]:
  $$\nabla_\mu S^D = -\frac{n(n-4)}{4}\nabla_\mu(\gamma_\nu\gamma^\nu) \ .$$

The paper is organised as follows : in the next Section, we recall the classification of Bianchi metrics and the expressions of geometrical objects, separating the 4-dimensional metric g into a "time part" and a 3-dimensional homogeneous one. Focussing ourselves on Class A Bianchi metrics, we exhibit a specific Gauduchon's gauge and show how, *in the diagonal case* the Einstein-Weyl equations simplify and ensure that the dual γ_μ of the Weyl one-form γ is a Killing vector, as in the compact case, and that the metric is either conformally Einstein or conformally Kähler. In particular, this proves that four-dimensional Einstein-Weyl spaces equipped with a diagonal Bianchi IX metrics are necessarily conformally Kähler, *i.e.* that our previous solution [GB] is the general one, up to a conformal transformation.
In Section 3, we show that for *all class A Bianchi metrics*, there exits a simple Gauduchon's gauge such that the conformal scalar curvature is constant on the manifold and the dual γ_μ of the Weyl one-form γ satisfies $D^\mu\left(\nabla_{(\mu}\gamma_{\nu)}\right) = 0$, where D denotes the covariant derivative with respect to the Weyl connection γ. Using these results, we prove that for Bianchi VI_0 , VII_0 , $VIII$, and IX, the most general solution of Einstein-Weyl constraints is the same as the one in the diagonal case, *i.e.* in the non-conformally Einstein cases, the Kähler one of previous subsection, up to a conformal transformation. Finally, we also prove that the only self-dual Einstein-Weyl structures are the Bianchi IX ones of Madsen [9, 13].

2 Bianchi metrics and Einstein-Weyl structures.

2.1 The geometrical setting

- A Weyl space [2] is a conformal manifold with a torsion-free connection D and a one-form γ such that for each representative metric g in a conformal class [g],

$$D_\mu g_{\nu\rho} = \gamma_\mu g_{\nu\rho} \ . \tag{1}$$

A different choice of representative metric : $g \longrightarrow \tilde{g} = e^f g$ is accompanied by a change in $\gamma \ : \gamma \longrightarrow \tilde{\gamma} = \gamma + df$. Conversely, if the one-form γ is exact, the metric g is conformally equivalent to a Riemannian metric $\tilde{g}$: $D_\mu \tilde{g}_{\nu\rho} = 0$. In that case, we shall speak of an *exact* Weyl structure.

- On the other hand, Bianchi metrics are real four-dimensional metrics with a three-dimensional isometry group, transitive on 3-surfaces. Their classification was done by Bianchi in 1897 according to the Lie algebras of their isometry group, *i.e.* according to the Lie algebra structure constants C^i_{jk} , $(i,\ j,\ k\ = 1,\ 2,\ 3\,)$; on general grounds, these ones may be decomposed into two parts [14]:

$$C^i_{jk} = n^{il}\epsilon_{jkl} + a_l[\delta^i_j\delta^l_k - \delta^i_k\delta^l_j] \tag{2}$$

where the symmetric 3×3 tensor n^{il} may be reduced to a diagonal matrix with entries 0,1 or -1 and the vector a_l satisfies

$$n^{il}a_l = 0\,.$$

This splits Bianchi metrics into two classes : class A in which the vector a_l is zero, and class B in which it has one non vanishing component, say a_1.

- An invariant Weyl struture may then be written as :

$$\begin{aligned} ds^2 = g_{\mu\nu}dx^\mu dx^\nu = dT^2 + h_{ij}(T)\sigma^i\sigma^j \quad &; \quad \gamma = \gamma_0(T)dT + \gamma_i(T)\sigma^i \ , \\ d\sigma^i = \frac{1}{2}C^i_{jk}\sigma^j\wedge\sigma^k \quad i,j,k = 1,2,3 \quad &; \quad \mu,\ \nu = (0,\alpha),\ (0,\beta)\ \ ;\ \sigma^i = \sigma^i_\alpha dx^\alpha \\ \sigma^i_\alpha\sigma^j_\beta h_{ij}(T) = g_{\alpha\beta} \quad &; \quad \sigma^\alpha_i\sigma^j_\alpha = \delta^j_i\ ,\ \sigma^\alpha_i\sigma^i_\beta = \delta^\alpha_\beta\ , \end{aligned} \tag{3}$$

where the three σ^i are one-forms invariant under the isometries of the homogeneous 3-space, characterised by the aforementionned structure constants C^i_{jk}. Notice that there is no loss of generality in choosing the metric element $g_{00} = 1$ as this corresponds to a choice of "proper time " T, but the matrix h_{ij} is *a priori* non-diagonal. On another hand, one might always choose a representative in the conformal class [g] such that $\gamma_0(T) \equiv 0$.

The Ricci tensor associated to the Weyl connection D is defined by :

$$[D_\mu,\, D_\nu\,]v^\rho = \mathcal{R}^{(D)\rho}{}_{\lambda,\mu\nu}\ v^\lambda \ \ ,\ \ \mathcal{R}^{(D)}_{\mu\nu} = \mathcal{R}^{(D)\rho}{}_{\mu,\rho\nu}\ . \tag{4}$$

$\mathcal{R}^{(D)}_{\mu\nu}$ is related to $R^{(\nabla)}_{\mu\nu}$, the Ricci tensor associated to the Levi-Civita connection [GB]:

$$\mathcal{R}^{(D)}_{\mu\nu} = R^{(\nabla)}_{\mu\nu} + \frac{3}{2}\nabla_\nu\gamma_\mu - \frac{1}{2}\nabla_\mu\gamma_\nu + \frac{1}{2}\gamma_\mu\gamma_\nu + \frac{1}{2}g_{\mu\nu}[\nabla_\rho\gamma^\rho - \gamma_\rho\gamma^\rho]\ . \tag{5}$$

$R^{(\nabla)}_{\mu\nu}$ may be expressed as (for exemple see [14]) :

$$\begin{aligned}
R^{(\nabla)}_{00} &= -\frac{1}{2}\frac{d}{dT}(\frac{h'}{h}) - \frac{1}{4}K^j_i K^i_j \ \ ; \ K^j_i = \frac{dh_{ik}}{dT}h^{kj} \ \ ; \ h = \det[h_{ij}] \ , h' = \frac{dh}{dT} \ , \\
R^{(\nabla)}_{0\alpha} &= \frac{1}{2}\sigma^k_\alpha[C^i_{jk} - \delta^i_k C^m_{mj}]K^j_i \ , \\
R^{(\nabla)}_{\alpha\beta} &= \sigma^i_\alpha \sigma^j_\beta \left[R^{(3)}_{ij} - \frac{1}{2}\frac{dK_{ij}}{dT} + \frac{1}{2}K^k_i K_{kj} - \frac{h'}{4h}K_{ij}\right] \ ; \ K_{ij} = K^k_i h_{kj} = \frac{dh_{ij}}{dT} \ , \text{ e.t.c...}
\end{aligned} \tag{6}$$

where $R^{(3)}_{ij}$, the 3-dimensional Ricci tensor associated to the homogeneous space Levi-Civita connection, in the basis of the one-forms σ^i, may be expressed as a function of the 3-metric h_{ij} and of the structure constants of the group [14, 15].
In the same way, the 4-dimensional Bianchi identity splits :

$$C^i_{jk}R^{(3)j}_i + C^i_{ji}R^{(3)j}_k = 0 \ \ , \ k = 1,2,3 \quad ([14], \ equ.(116,26)) \tag{7}$$

and (see the appendix) :

$$h^{ij}\frac{d}{dT}R^{(3)}_{ij} \equiv \frac{dR^{(3)}}{dT} + K^j_i R^{(3)i}_j = 2C^i_{ji}R^j_0 \ \ , \text{ with } R^j_0 = h^{ji}\sigma^\alpha_i R^{(\nabla)}_{0\alpha} \ \ , \ R^{(3)} = R^{(3)}_{ij}h^{ij} \ . \tag{8}$$

We do not find equation (8) in the standard textbooks on gravity.

2.2 The Gauduchon's gauges

We computed (using equations (39,40 of the appendix) the components of the tensor $\nabla_{(\mu}\gamma_{\nu)}$ and find ;

$$\begin{aligned}
\nabla_0\gamma_0 &= \frac{d\gamma_0}{dT} \\
\nabla_{(0}\gamma_{\alpha)} &= \frac{1}{2}\sigma^i_\alpha h_{ij}\frac{d\gamma^j}{dT} \\
\nabla_{(\alpha}\gamma_{\beta)} &= \frac{1}{2}\sigma^i_{(\alpha}\sigma^j_{\beta)}[\gamma_0 K_{ij} + 2\gamma^k h_{il}C^l_{kj}] \ ,
\end{aligned} \tag{9}$$

and, as a consequence,

$$\nabla_\mu\gamma^\mu = \frac{1}{\sqrt{h}}\frac{d}{dT}[\sqrt{h}\gamma_0] - C^i_{ij}\gamma^j \ . \tag{10}$$

When $C^i_{ij} \equiv 2a_j = 0$, which corresponds to class A Bianchi metrics, a special Gauduchon's gauge is obtained through the choice :

$$\gamma_0(T) \equiv 0\,. \tag{11}$$

In the compact case, the choice (11) is the unique good one ([9],Proposition 5.20).

2.3 The Einstein-Weyl equations

Einstein-Weyl spaces are defined by :

$$\begin{aligned}
\mathcal{R}^{(D)}_{(\mu\nu)} &= \Lambda' g_{\mu\nu} \quad \Leftrightarrow \\
R^{(\nabla)}_{\mu\nu} + \nabla_{(\mu}\gamma_{\nu)} + \frac{1}{2}\gamma_\mu\gamma_\nu &= \Lambda \ g_{\mu\nu} \ \ , \ \ \Lambda = \Lambda' - \frac{1}{2}[\nabla_\lambda\gamma^\lambda - \gamma_\lambda\gamma^\lambda] \ .
\end{aligned} \tag{12}$$

Note that for an exact Einstein-Weyl structure, $\gamma = df$, the representative metric is conformally Einstein. Note also that the conformal scalar curvature is related to the scalar curvature through:

$$S^{(D)} = g^{\mu\nu}\mathcal{R}^{(D)}_{\mu\nu} = 4\Lambda + 2[\nabla_\lambda \gamma^\lambda - \gamma_\lambda \gamma^\lambda] = R^{(\nabla)} + 3[\nabla_\lambda \gamma^\lambda - \frac{1}{2}\gamma_\lambda \gamma^\lambda] \ . \tag{13}$$

For Class A Bianchi metrics, in the special Gauduchon's gauge (11), Einstein-Weyl constraints (12) splits into :

$$\Lambda = -\frac{1}{2}\frac{d}{dT}(\frac{h'}{h}) - \frac{1}{4}K_i^j K_j^i \ \ , \tag{14}$$

$$n^{ij}\epsilon_{jkl}K_i^k = -h_{li}\frac{d\gamma^i}{dT} \ , \tag{15}$$

$$\Lambda h_{ij} = R^{(3)}_{ij} - \frac{1}{2}\frac{dK_{ij}}{dT} + \frac{1}{2}K_i^k K_{kj} - \frac{h'}{4h}K_{ij} + \frac{1}{2}\gamma_i\gamma_j + \frac{1}{2}\gamma^k[h_{im}n^{mn}\epsilon_{nkj} + h_{jm}n^{mn}\epsilon_{nki}] \ . \tag{16}$$

2.4 Diagonal metrics and conformal Kählerness

Let us restrict ourselves to the diagonal Bianchi metrics, usually written as [7, 8]:

$$ds^2 = \omega_1\omega_2\omega_3(dt)^2 + \frac{\omega_2\omega_3}{\omega_1}(\sigma^1)^2 + \frac{\omega_1\omega_3}{\omega_2}(\sigma^2)^2 + \frac{\omega_1\omega_2}{\omega_3}(\sigma^3)^2 \tag{17}$$

Define α_i through :

$$\frac{d\omega_i}{dt} = \alpha_i\omega_i + n^{ii}\omega_j\omega_k \quad , \quad (i,j,k) = \text{circ. perm. } (1,2,3) \ . \tag{18}$$

In [7], Dancer and Strachan gave the conditions on the α_i under which the four dimensional diagonal Bianchi metric is Kähler, but not Hyper-Kähler. These conditions are :
- Class A : two of the α_i have to be equal, the third one vanishing ;
- Class B : the three α_i have to be proportional to ω_1 and to satisfy : $\alpha_1 = \alpha_2 + \alpha_3$.

Under a conformal transformation preserving the cohomogeneity-one character of a Bianchi metric : $\tilde{g} = \mu^2(T)g$, these conditions are easily converted into conditions for Kählerness up to a conformal transformation :
Lemma 1 : *A diagonal Bianchi metric (17) is conformal to a Kähler one iff. :*

- *Class A metric ($a_i = 0$) : two of the α_i are equal* ;
- *Class B metric ($a_i = a\delta_{i1}$) : the following relations hold* :

$$\frac{\alpha_1 - \alpha_3}{a\omega_1} = \frac{a\omega_1}{\alpha_2 - \alpha_1} = \text{Cste} \ .$$

For a Class A diagonal Bianchi metric , equation (15) leads to

$$\gamma^i(T) = \Gamma^i \text{ constant} \ , \tag{19}$$

and equations (16) wrote for $i \neq j$:

$$\begin{gathered} \frac{1}{2}\gamma_i\gamma_i + \nabla_{(i}\gamma_{j)} = 0 \quad , \quad i \neq j \quad \Leftrightarrow \\ \omega_1\omega_2\omega_3\Gamma^i\Gamma^j = \Gamma^k[n^{ii}\omega_j^2 - n^{jj}\omega_i^2] \quad , \quad (i,j,k) = \text{circ. perm. } (1,2,3) \ . \end{gathered} \tag{20}$$

By inspection of the different possibilities for the n^{ii} [14], it is readily shown that at least two of the Γ^i necessarily vanish, with no other constraint for Bianchi I and II ; for Bianchi VI_0, the three of them vanishing, the metric is necessarily conformally Einstein ; for Bianchi VII_0 and $VIII$ $[n^{11}=n^{22}=+1]$ the third Γ^3 is constrained by

$$\Gamma^3[\omega_1^2-\omega_2^2]=0\ ,$$

then, either the metric is conformally Einstein or, with $\omega_1^2=\omega_2^2$, the metric is conformally Kähler (thanks to Lemma 1). For Bianchi IX case, the same result holds, the special direction being unfixed (it will be chosen in the same direction as for Bianchi VII_0 and $VIII$). A Corollary of this analysis is that in all Class A cases, the dual of the one form γ is a Killing vector.

In these three types of Bianchi metrics,

$$n^{11}=n^{22}=+1\quad,\quad\Gamma^1=\Gamma^2=0\quad,\quad\omega_1=\omega_2=\omega\quad,\quad\alpha_1=\alpha_2=\alpha\quad,$$

and the remaining equations (14,16) wrote in the vierbein basis corresponding to (17)(a comma indicates a derivative with respect to t):

$$\begin{aligned}
{\scriptstyle(00)}\quad 2(\omega)^2\omega_3\Lambda &= -2\alpha'-\alpha_3'-(\alpha_3)^2+4\alpha_3\alpha+2\alpha_3\omega_3+n^{33}\frac{(\omega)^2}{\omega_3}(2\alpha-\alpha_3)\\
{\scriptstyle(11,22)}\quad 2(\omega)^2\omega_3\Lambda &= -\alpha_3'-n^{33}\frac{(\omega)^2}{\omega_3}(2\alpha-\alpha_3)\\
{\scriptstyle(33)}\quad 2(\omega)^2\omega_3\Lambda &= (\Gamma^3)^2\omega^4-2\alpha'+\alpha_3'-2\alpha_3\omega_3+n^{33}\frac{(\omega)^2}{\omega_3}(2\alpha-\alpha_3)
\end{aligned}\tag{21}$$

Consider the function $u(t)=\frac{\alpha_3}{\omega^2}$. Its derivative is readily obtained, using the difference of the (00) and (33) equations (21) :

$$\frac{du}{dt}=-\frac{1}{2}\omega^2[(\Gamma^3)^2+u^2]\quad<\ 0\,.$$

Then one can change the variable t into u and compute :

$$\frac{d\omega_3}{du}=-2\frac{n^{33}+u\omega_3}{(\Gamma^3)^2+u^2}\quad,$$

which integrates to :

$$\omega_3(u)=2\frac{-n^{33}u+k}{(\Gamma^3)^2+u^2}\,.\tag{22}$$

Then one gets :

$$\alpha(u)=-2\frac{-n^{33}u+k}{(\Gamma^3)^2+u^2}-\frac{1}{4}[(\Gamma^3)^2+u^2]\frac{d\omega^2}{du}\quad,\quad\alpha_3(u)=u\omega^2(u)\,.\tag{23}$$

The difference of the (11) and (33) equations (21) then gives a second order linear differential equation on $\omega^2(u)$:

$$\begin{aligned}
\frac{d^2\omega^2}{du^2}\quad&+\quad\left[\frac{6u}{(\Gamma^3)^2+u^2}-\frac{2n^{33}}{n^{33}u-k}\right]\frac{d\omega^2}{du}\\
&-\quad 4\left[\frac{(\Gamma^3)^2}{((\Gamma^3)^2+u^2)^2}+\frac{k}{(n^{33}u-k)((\Gamma^3)^2+u^2)}\right]\omega^2+\frac{8n^{33}}{((\Gamma^3)^2+u^2)^2}=0\,.
\end{aligned}\tag{24}$$

The solution is :

$$\begin{aligned} \omega^2 &= \frac{4}{(\Gamma^3)^2+u^2}\Omega^2 \ , \ \text{with} \quad \Omega^2 = n^{33} + \lambda_1[n^{33}(u^2-(\Gamma^3)^2)-2ku] + \\ &+ \lambda_2\Big[[n^{33}(u^2-(\Gamma^3)^2)-2ku]\Gamma^3\arctan(\frac{u}{\Gamma^3}) + (\Gamma^3)^2[n^{33}u-2k]\Big] \ . \end{aligned} \tag{25}$$

Equations (22,25) and

$$\frac{du}{dt} = -2\Omega^2 \tag{26}$$

give the distance [1] and Weyl form as functions of the new "proper time" u :

$$\begin{aligned} ds^2 &= 2\frac{-n^{33}u+k}{\Omega^2((\Gamma^3)^2+u^2)^2}(du)^2 + 2\frac{-n^{33}u+k}{(\Gamma^3)^2+u^2}[(\sigma^1)^2+(\sigma^2)^2] + 2\frac{\Omega^2}{-n^{33}u+k}(\sigma^3)^2 \ , \\ \gamma &= \frac{2\Gamma^3\Omega^2}{-n^{33}u+k}\sigma^3 \ . \end{aligned} \tag{27}$$

Finaly, the conformal scalar curvature is the constant

$$S^D = 4\lambda_2(\Gamma^3)^4 \ . \tag{28}$$

Under the conformal transformation $\tilde{g} = [(\Gamma^3)^2+u^2]g/2$, the metric may be rewriten in the standard form (17) with

$$\tilde{\omega}_1 = \tilde{\omega}_2 = \Omega\sqrt{(\Gamma^3)^2+u^2)} \ , \quad \tilde{\omega}_3 = -n^{33}u+k \, ,$$

the "proper time" $\tilde{t}$ being given by

$$d\tilde{t} = -\frac{du}{\Omega^2((\Gamma^3)^2+u^2)} \, .$$

Then,

$$\frac{d\tilde{\omega}_3}{d\tilde{t}} - n^{33}\tilde{\omega}_1\tilde{\omega}_2 = \tilde{\omega}_3\tilde{\alpha}_3 = 0 \quad , \tilde{\alpha}_1 = \tilde{\alpha}_2 \ ,$$

ensuring that the metric $\tilde{g}$ is Kähler.

Then we have proved the

Theorem 1 : *The most general (non-)compact non-exact Einstein-Weyl structure with a diagonal Bianchi VII_0, $VIII$ or IX metric is conformal to a Kähler 4-parameters's one. In particular, in the Bianchi IX case, the Kähler metric is the one found in [GB, equ.(27)].*

In the following Section, we shall consider non-diagonal Bianchi metrics [2] , but still restrict ourselves to Class A ones, where the particular choice of Gauduchon's gauge (11) will be of great help.

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[1] Of course, the 4 parameters k, Γ^3, λ_1, λ_2 and the "time" variable u are constrained by positivity : $\Omega^2 > 0$, $-n^{33}u+k > 0$.

[2] When $\gamma = 0$ (Einstein equations), and for Bianchi $VIII$ and IX metrics, it was shown in [8] that, thanks to (15), the looked-for Einstein metrics may be chosen to be diagonal. I thank Paul Tod for a clarifying discussion on that assertion.

3 (Non-)compact Einstein-Weyl structures with class A Bianchi metrics.

We first prove the
Lemma 2 : *In the special gauge $\gamma_0 = 0$, Einstein-Weyl structures with a Class A Bianchi metric have a constant conformal scalar curvature S^D.*

Using $\nabla_\mu \gamma^\mu = 0$, the conformal scalar curvature (13) writes

$$S^{(D)} = 4\Lambda - 2\gamma_i \gamma^i \ . \tag{29}$$

Contracting equation (16) with K^{ij}, using (14,15), leads to :

$$K^{ij} R^{(3)}_{ij} = \frac{d}{dT}\left[\frac{1}{4}K^j_i K^i_j - (\frac{h'}{2h})^2 - \frac{1}{2}\gamma_i\gamma^i\right] \ . \tag{30}$$

Our Bianchi identity (8), with $C^i_{ij} = 2a_j = 0$, then gives :

$$R^{(3)} + \frac{1}{4}K^j_i K^i_j - (\frac{h'}{2h})^2 - \frac{1}{2}\gamma_i\gamma^i = \text{ Constant} \ . \tag{31}$$

Contracting now equation (16) with h^{ij}, using (14), leads to :

$$R^{(3)} + \frac{3}{4}K^j_i K^i_j + \frac{d}{dT}(\frac{h'}{h}) - (\frac{h'}{2h})^2 + \frac{1}{2}\gamma_i\gamma^i = 0 \tag{32}$$

which, combined with (31,14) gives

$$2\Lambda - \gamma_i\gamma^i \equiv S^{(D)}/2 = \text{ Constant} \qquad \text{Q.E.D.} \tag{33}$$

We have the
Corollary 1 : *Einstein-Weyl structures with a Class A Bianchi metric have a conformal scalar curvature $S^{(D)}$ of constant sign on the manifold.*

We may now prove the
Lemma 3 : *In any Gauduchon's gauge $\nabla_\mu \gamma^\mu = 0$, all Einstein-Weyl structures with a constant conformal scalar curvature S^D are such that $D^\mu[\nabla_{(\mu}\gamma_{\nu)}]$ vanishes.*

Acting with ∇^μ on the Einstein-Weyl constraint (12) in the Gauduchon gauge and using the four-dimensional Bianchi identity, the constant value of $S^D \equiv R^{(\nabla)} - 3/2\gamma_\mu\gamma^\mu$, one gets :

$$\nabla^\mu[\nabla_{(\mu}\gamma_{\nu)}] + \gamma^\mu[\nabla_{(\mu}\gamma_{\nu)}] = -\frac{1}{4}\nabla_\nu S^D \qquad Q.E.D. \tag{34}$$

Note that in the compact case, contraction of the previous identity with γ^ν, followed by an integration on the manifold, leads to the vanishing of $\nabla_{(\mu}\gamma_{\nu)}$ [11].

- Considering the $\nu = 0$ component of the previous equation, the expression of $[\nabla_{(\mu}\gamma_{\nu)}]$ given by (9), and the formula (41) given in the appendix, we obtain for any class A Bianchi metric :

$$h_{ij}\frac{d}{dT}(\frac{1}{2}\gamma^i\gamma^j) = 0 \ . \tag{35}$$

- In the same way, considering the $\nu = \alpha$ component of the equation (34), and multiplying by σ^α_i gives after some manipulations [using the expression of $\nabla_\alpha \sigma^i_\beta$ given in the appendix (40)]:
$$\frac{d}{dT}(h_{ij}\frac{d}{dT}(\gamma^j)) + \frac{h'}{2h}(h_{ij}\frac{d}{dT}(\gamma^j)) = C^j_{ik}\gamma_j\gamma^k + X_{ij}[h_{mn}]\gamma^j \ , \tag{36}$$
where the 3×3 symmetric matrix $X_{ij}[h_{mn}]$ is given by
$$X_{ij} = h_{mn}h^{pq}C^m_{pi}C^n_{qj} + C^m_{ni}C^n_{mj} \ ,$$
which may be expressed for a Class A Bianchi metric as:
$$X_{ij} = \frac{1}{\det h_{mn}}[Tr(hnhn)h_{ij} - (hnhnh)_{ij}] + \epsilon_{ikr}\epsilon_{jls}n^{kl}n^{rs} \, . \tag{37}$$

The contraction of (36) by γ^i and the use of (35), finally gives :
$$\frac{d\gamma^i}{dT}h_{ij}\frac{d\gamma^j}{dT} + \gamma^i X_{ij}\gamma^j = 0 \, . \tag{38}$$

Then we have:
Lemma 4 : *For any Class A Bianchi metric h_{ij} such that $(\gamma,\ h)$ is an Einstein-Weyl structure, the Weyl form γ may be written in our particular Gauduchon's gauge as : $\gamma = \Gamma^i h_{ij}(T)\sigma^j$, where the Γ^i are constant parameters.*

Indeed, at any given time T one can find coordinates $\tilde{\sigma}^i$ such that h_{ij} is a diagonal matrix $\tilde{h}_{ij}$, the structure constants being unchanged. The matrix $\tilde{X}$ is then diagonal too, with elements
$$\tilde{X}_{11} = [(n^{22}\tilde{h}_{22} - n^{33}\tilde{h}_{33})^2]/(\tilde{h}_{22}\tilde{h}_{33})$$
and circular permutations. h_{ij} being a strictly positive definite matrix, we get the vanishing of $\frac{d\tilde{\gamma}^i}{dT}$, and, at that time of $\frac{d\gamma^j}{dT}$ in any coordinate frame ; the same results then holds at any proper time. Q.E.D.

We are now in position to discuss the issue of the diagonal hypothesis for the metric $h_{ij}(T)$. In the Einstein equation analysis, as explained by Tod [8], the condition (15, with $\gamma_i = 0$) ensures - at least for Bianchi IX and $VIII$ cases [3] -, a possible simultaneous diagonalisation of the matrices h_{ij} and $\frac{dh_{ij}}{dT}$ or K_{ij} at T_0 , with no change of the structure constants n^{ij} .

Here [4], let us start from a proper time T_0 such that h_{ij} (and n^{ij}) is diagonal. By inspection of the possible values of n^{ii} , equation (38) ensures that the value of the constants Γ^i fall into one of three cases :

- all zero : in particular, this is the sole solution in the Bianchi VI_0 case. In such a situation, there exists no non-exact Einstein-Weyl structure, and Tod's argument ensures that for Bianchi VI_0 , VII_0 , $VIII$ and IX cases, there is no loss of generality in the choice of a diagonal metric $h_{ij}(T)$.

[3] As a matter of facts, Tod's argument can be also used in Bianchi VI_0 and VII_0 cases.

[4] We leave aside the Bianchi I case, with vanishing structure constants and matrix X, constant one-form coefficients Γ^i and where, locally, $\sigma^i \equiv dx^i$. Two of the Γ^i vanish, but one cannot prove that the metric stay in a diagonal form.

- at most one of them vanishes : this may happen only in the Bianchi IX case with $\tilde{h}_{ij}(T_0) = h_0\delta_{ij}$. Then, a possible simultaneous diagonalisation of the matrix $\frac{d\tilde{h}_{ij}}{dT}$ or $\tilde{K}_{ij}$ at T_0 is possible, and (38), at $T = T_0 + \epsilon$, enforces the equalness of the $\tilde{\tilde{K}}_{ii}$ at T_0 . So, at that time, the matrices $\tilde{\tilde{n}}$, $\tilde{\tilde{h}}$, $\tilde{\tilde{K}}$ are proportional to the 3×3 unit matrix. Then, one can find new coordinates where $\frac{d\tilde{\tilde{K}}}{dT}$ is also diagonal, which ensures that the matrices h and K stay in a diagonal form. But, equation (16), where the term $\tilde{\tilde{\gamma}}_i\tilde{\tilde{\gamma}}_j$ is not in a diagonal form, contradicts the hypothesis of at most one of the Γ^i vanishing.

- one of them subsists :

  – this is the case for Bianchi II case, but (15) enforces no further constraint on the metric and it seems hard to prove that the metric will stay in a diagonal form ;

  – this occurs in Bianchi VII_0 , $VIII$ and IX cases, when at that time, one of the $\tilde{X}_{ii}$ given previously vanishes, say $\tilde{X}_{33}$. For these three cases, we have, for a non-exact Eintein-Weyl structure:

$$\tilde{n}^{11} = \tilde{n}^{22} = +1 \quad , \quad \tilde{\Gamma}^1 = \tilde{\Gamma}^2 = 0 \ , \quad \tilde{\Gamma}^3 \neq 0 \ , \quad \tilde{h}_{11}(T_0) = \tilde{h}_{22}(T_0) \, .$$

  Condition (15) ensures that at T_0 :

$$\frac{d\tilde{h}_{31}}{dT} = \frac{d\tilde{h}_{32}}{dT} = 0 \, .$$

  As a consequence, the particular block diagonal structure of the matrices $\tilde{h}_{ij}$, $\tilde{n}^{ij}$ and $\frac{d\tilde{h}_{ij}}{dT}$ ensures that they may be simultaneously diagonalised at T_0 . So $\tilde{\tilde{h}}_{ij}$ and $\tilde{\tilde{K}}_{ij}$ (thanks to equ. (16)) stay diagonal and we have proved that the constraints that result from Einstein-Weyl equations for Bianchi IX, $VIII$ and VII_0 in the non-diagonal case, are the same as the ones in the diagonal situation.

We can summarize this discussion in a theorem :
Theorem 2 : *(Non-)compact Einstein-Weyl Bianchi metrics of the types VI_0, VII_0, $VIII$ and IX are conformally Kähler or conformally Einstein and the metric may be taken in a diagonal form. In the non-exact Einstein-Weyl case, the metric and Weyl form were given in equ.(27). The conformal scalar curvature has a constant sign on the manifold and, in our particular Gauduchon's gauge, the dual of the Weyl form is a Killing vector.*

Theorem 1 then gives the following :
Corollary 2 : *(Non-)compact non-exact Einstein-Weyl Bianchi IX metrics are conformally Kähler. The metric may be taken in a diagonal form and is conformal to the 4-parameters' one given in [GB.equ.(27)].*

4 Concluding remarks

In this paper, we have presented a (nearly) complete analysis of the Einstein-Weyl structures $(g\, ,\, \gamma)$ corresponding to Class A Bianchi metrics. We have shown that, also in the non-compact case, there exists a conformal gauge in which the conformal scalar curvature is a constant, and we have proved that types VI_0 , VII , $VIII$ and IX, diagonal or not, are conformally Kähler or

conformally Einstein. We have explained why, in these cases, one can restrict oneself to diagonal metrics. Moreover, in the non-exact Eintein-Weyl cases, the explicit expression for the distance and Weyl 1-form, depending on 4 parameters submitted to some positivity requirements has also been obtained in subsection 2.4.

The further requirement of completeness and compactness will restrict the parameters of our solutions : in particular, Bianchi VI_0, VII and $VIII$ metrics cannot give compact metrics, their isometry group being non-compact. We shall give elsewhere the full family of Compact Bianchi IX Einstein-Weyl metrics, which, as we have proven here, are conformally Kähler [16].

Let us make a final comment on self-duality constraints on the Weyl connection γ. In the vierbein basis corresponding to expression (27), one obtains

$$d\gamma = \Gamma^3((\Gamma^3)^2 + u^2)\left[\frac{d}{du}[\frac{\Omega^2}{-n^{33}u+k}]e^0\wedge e^3 + n^{33}\frac{\Omega^2}{(-n^{33}u+k)^2}e^1\wedge e^2\right] .$$

The (anti-)self duality of the Weyl connection then needs

$$\Omega^2 = C(-n^{33}u+k)^{1\pm 1} .$$

Due to positivity requirements on Ω^2, solutions exist only in the Bianchi IX case, and were given in [GB.Corollary 3][2, 13].

5 Appendix

Using equations (3) and the definition of K_i^j given in (6), the Christoffel connection components are expressed as :

$$\begin{aligned} 2\Gamma^\alpha_{0\beta} &= \sigma_i^\alpha\sigma_\beta^j K_j^i \quad , \quad 2\Gamma^0_{\alpha\beta} = -\sigma^i_\alpha\sigma^j_\beta K_{ij} \ , \\ 2\Gamma^\alpha_{\beta\gamma} &= g^{\alpha\delta}[g_{\delta\beta\,,\gamma} + g_{\delta\gamma\,,\beta} - g_{\beta\gamma\,,\delta}] \quad , \quad \text{the other components vanishing .} \end{aligned} \tag{39}$$

Then, the covariant derivative of the three basis vectors σ^i_α are found to be :

$$\nabla_\alpha\sigma^i_\beta = \partial_\alpha\sigma^i_\beta - \Gamma^{(3)\gamma}_{\alpha\beta}\sigma^i_\gamma = \frac{1}{2}C^i_{jk}\sigma^j_\alpha\sigma^k_\beta + h^{ij}h_{kl}C^k_{jn}\sigma^l_{(\alpha}\sigma^n_{\beta)} . \tag{40}$$

The expression

$$K_i^j\sigma_j^\beta\nabla_\beta\sigma^i_\alpha = C^i_{jk}K^j_i\sigma^k_\alpha$$

will be useful, as well as

$$\nabla_\alpha\sigma^\alpha_i = -\sigma^\beta_i\sigma^\alpha_j\nabla_\alpha\sigma^j_\beta = C^j_{ij} . \tag{41}$$

The $\nu = 0$ component of the Bianchi identity $2\nabla_\mu R^{(\nabla)\,\mu}_\nu = \nabla_\nu R^{(\nabla)}$ is split according to $\mu = (0\,,\alpha)$. Using (6,40) and

$$R^{(\nabla)} = R^{(3)} + 2R^{(\nabla)}_{00} - \frac{1}{4}(\frac{h'}{h})^2 + \frac{1}{4}K_{ij}K^{ij}$$

one obtains :

$$2\nabla_\mu R^{(\nabla)\,\mu}_0 = \nabla_0 R^{(\nabla)} - h^{ij}\frac{dR^{(3)}_{ij}}{dT} + 2\nabla^{(3)}_\alpha R^{(\nabla)\,\alpha}_0 .$$

As a consequence :

$$h^{ij}\frac{dR^{(3)}_{ij}}{dT} = 2[\nabla^{(3)}_\alpha\sigma^\alpha_k]R^k_0 , \tag{42}$$

where $R^k_0 = h^{ij}\sigma^\alpha_i R^{(\nabla)}_{0\alpha}$.

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